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The use of cumulative sum control charts for testing product durability and reliability

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Abstract:

Aim: The aim of the paper is to test and confirm the thesis on the possibility of the application of the Cumulative Sum Control Charts (CUSUM Charts) effectively in monitoring reliability and durability of a product.

Design/Research method: The study employed the cumulative sum control procedures that can be used for diagnostic variables distributed exponentially. This kind of distribution is often used to monitor reliability and durability of products or processes.

Conclusions/Findings: The sequential methods presented and examples of applications of the solutions outlined in the paper seem to confirm the validity of the thesis on the possibility of using effectively the CUSUM charts for monitoring product reliability and durability. The methods presented in the paper can be used in practice by economists, quality managers or production technologists.

Originality/Value of the article: The novelty (originality) of the paper lies in that the control charts presented can be used to either accept or disqualify the process analyzed. This feature is not present in the classic CUSUM procedures. This paper is a part of the wider collection of the author's works focusing on the problems surrounding acceptance cusum control charts. According to the author's knowledge, no similar studies have as yet been conducted.

Keywords: reliability of products, control charts, cumulative sum (cusum) control charts
JEL: C12, C19, C44, L15

1. Introduction

The concepts of durability and reliability are important issues associated with the concept of the quality of a product. Already W.E. Deming (1900-1993) in his definition of quality

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maintains that it is "a predictable *degree of uniformity* and dependability at low cost and suited to the market" (Deming 1982: 1-2). In the standard, (PN ISO 9000 2000), we come across a rather superficial definition of reliability according to which "reliability is a general term used to describe availability of an object and factors affecting it: reliability, maintainability, and ensuring means of operations". A more precise definition of reliability and at the same time durability of a product is given by Dobiesław Bobrowski (Bobrowski 1985: 7). The author uses a more general concept of reliability with respect to a technical item. According to him, "A mathematical model of a non-renewable technical item which describes its reliability, i.e. predictable ability to perform operations under specific conditions and time interval is a nonnegative continuous random variable T called a *useful life* or *durability* of an item".

The concept of reliability is also related to that of durability. In the work of Iwasiewicz (Iwasiewicz 2005: 30) one can come across a definition stating that „durability of a product is the period (time interval) in which it does not show any substantial reduction in the initial level of its quality under specific conditions of its operation (use) or storage". The same work cites the following definition of reliability: "It is the probability that the evaluated product that is used in specific conditions will maintain its capability to meet the demands posed on it over a specific period of its useful life". Thus, durability is part of reliability. A high durability and reliability of a product help it to achieve a high quality. However, it should be specified that a high reliability is not yet a guarantee that a product will also be of high quality, as it merely contributes to its quality, though in a substantial way.

One of the important concepts related to reliability of products seems to be the strength of materials from which products are manufactured. The subject matter of the science concerned with strength of materials is the evaluation of durability of typical construction elements which are subject to forces acting on them (Pietrzakowski 2010: 10). Sometimes, testing strength is combined with time measurement. For example, while assessing the strength of a towing rope it is assumed that it should withstand a tensile load applied with a specific force for at least 20 seconds.

The basic measure of the product's (item's) reliability in a time interval $[0, t]$ is the probability which, in short, is called reliability or a reliability function:

$$R(t) = \mathrm{P}(T \geq t) \quad (t \geq 0), \qquad (1)$$

where: $R(t)$ – reliability function,

T – a random variable, which is the time during which a product (item) works properly

A similar convention is also used to define the failure function *F (t):*

$$F(t) = P(T < t) = 1 - R(t), \qquad (2)$$

which can be viewed as the distribution function of a random variable T at point t.

Sometimes, instead of using the symbol T (t) (time) in the formulas (1), (2), a different symbol can be applied, e.g. X (x) denoting, e.g. the number of repetitions of a specific action. This refers, for example, to products which are used cyclically or periodically. Then, the variable X denotes, e.g. the number of on and off switchings of an electricity switch, the number of opening and closing a door, drawer, etc. A similar situation occurs when measuring the so called ultimate strength before a towing rope breaks. In measuring this kind of strength, the assumption is that the rope should withstand a load greater than the one given by the relevant regulations and manufacturer's declaration. The reliability of the rope should then be understood as the probability that it will not break when applied with a force that is above or equal to x [1].

The random variable T (or X) is usually subject to an exponential distribution. However, it should be specified that the exponential distribution is not the only distribution that can be used in the evaluation process of product durability or reliability. In the literature (see e.g. Macha 2001: 75-92; Oltanu 2010: 4-12; Walanus 2000: 51-59) Weibull distribution or lognormal distribution is used. It is relatively less frequently that a normal distribution or uniform distribution is applied. Weibull distribution is more general than the exponential distribution. It is employed when the failure rate is a variable whose plot is monotonic. This distribution is used to describe fatigue service life of materials and mechanical structures. The lognormal distribution, on the other hand, can be applied for testing reliability of objects whose failures are caused by steadily increasing fatigue cracks or corrosion. By generalizing the exponential distribution and Weibull distribution, we can obtain gamma distribution, which, with specific assumptions, can become the following distributions: Reley, Maxwell, Erang or Chi-square distribution. The gamma distribution plays a special role in modeling reliability of renewable products[2].

Considering the paper's limited space, it is assumed further on that the random variable

[1] An example presented in the further section of the paper is concerned with the issue of testing reliability of towing ropes

[2] See more in e.g. Macha (2001: 107-110), Walanus (2000: 51-59), Kowalski (2010).

analyzed has an exponential distribution.

This opens up a variety of possibilities in terms of tracking and actively analyzing reliability based on the diverse statistical procedures. These include, among other things, Page's cumulative sum control charts (Page 1954a: 100-115; 1954b: 136-139), which draw on the sequential testing of statistical hypotheses for which we have to thank Abraham Wald (1902-1950) (Wald 1945, 1947). Cumulative sum procedures have been described time and again in the worldwide and Polish scholarly literature. Attention should be paid to, for example, such contemporary works as: Montgomery (2009: 400-419), Qiu P. (2013: 119-180), White et al. (1997: 673-679), Hawkins et al. (1998), Lucas (1985: 129-144), Lowry (1995: 409-431), Gan (1993: 445-460). Moreover, among the works published in Poland, one should mention: Iwasiewicz et al. (1988), Iwasiewicz (1999: 243-268), Kończak (2007: 94-97), Thompson et al. (2005: 149-196), Sałaciński (2009: 60-67), Czabak-Górska (2017: 281-290), Bartkowiak (2011: 63-71), Sankle et al. (2012: 95-106). Constraining the topic of research to the implementation of cumulative sum procedures while exploring durability or reliability of products, one should mention such works as, for example: Olteanu (2010), Shafae et al. (2015: 839-849), Rao (2013: 229-234), Sze, Pascual (2013: 1-33), Zhang et al. (2012: 275-286).

The majority of studies within this scope presents tools which are based on the classic construction of cumulative sum control charts. Therefore, they allow the alternative hypothesis that a product does not meet the quality requirements linked to the level of its reliability to be accepted only at a specific risk of error α. Moreover, they do not allow the null hypothesis to be accepted, according to which the product meets the quality requirements as regards its reliability level. This is made possible by e.g. the cumulative sum control procedures with an option of the process acceptance, which are a modified form of classic cumulative sum control charts.

The aim of the paper is to conduct tests and analyses which may confirm the thesis that cumulative sum control charts can be used to monitor reliability of products/processes. The paper presents a modified cumulative sum procedure which allows the quality of products to be controlled in terms of diagnostic variables distributed exponentially. This study is part of the larger collection of the author's works concerned with the problems of acceptance control charts, among which one should mention: Major (1997, 2015a, 2015b, 2016). Moreover, the content of this paper follows up and builds on the works of Iwasiewicz (2008-2009; 2011).

2. Product quality control in terms of diagnostic variables distributed exponentially

As noted in the previous section, exponential distribution is most commonly used to describe reliability of non-repairable elements of technical systems (see Iwasiewicz et al. 1988: 90). We therefore say that the variable T is distributed exponentially with parameter λ ($T \sim W(\lambda)$), fulfilling the following properties:

$$E(T) = 1/\lambda, \text{ and } D^2(T) = 1/\lambda^2, \qquad (3)$$

where $E(T)$ and $D^2(T)$ denote, respectively the expected value and variance of variable T.

The tolerance interval of variable T is usually limited on the left-hand side by value t_d, while the range of non-target values is $(0, t_d)$, and that of target values is $(t_d, +\infty)$. The hypotheses to be tested are in the following form (after Iwasiewicz et al. 1988: 90):

$$H_0: Q_T = Q_0 \qquad (4)$$

$$H_{-1}: Q_T = Q_{-1} \quad (Q_{-1} < Q_0) \qquad (5)$$

where:

$Q_T = E(T) = 1/\lambda$,

Q_0, Q_{-1} values determined based on technical and economic reasons, such that $Q_{-1} < Q_0$

Hypotheses (4) and (5) can also be given by:

$$H_0: \lambda = \lambda_0 \qquad (6)$$

$$H_{-1}: \lambda = \lambda_1 \quad (\lambda_1 > \lambda_0) \qquad (7)$$

where

$\lambda_0 = 1/Q_0$ i $\lambda_1 = 1/Q_{-1}$ and $\lambda_1 > \lambda_0$.

It is assumed that the product fulfills quality requirements if $Q_T \geq Q_0$ ($\lambda \leq \lambda_0$). On the other hand, if: $Q_T \leq Q_{-1}$ ($\lambda \geq \lambda_1$), then the product fails to fulfill these requirements. Partial failure p_T is equivalent to the failure function (2), partial correctness q_T is the same as reliability function (1). Suppose that diagnostic variable T is exponentially distributed with a parameter λ ($T \sim W(\lambda)$), the tolerance interval T with the left-handed limit of t_d ($t_d \geq 0$), the equations (2) and (1) can be written a follows:

$$p_T = P(T < t_d) = F(t_d) = 1 - e^{-\lambda t_d}, \qquad (8)$$

$$q_T = 1 - p_T = P(T \geq t_d) = 1 - F(t_d) = e^{-\lambda t_d} = R(t_d). \qquad (9)$$

3. Classical sequential procedure[3]

A sequential analysis consists in a random sampling of individual or small sets of statistical population and finding each time whether the information thus far gathered allows for making a specific decision. With the null hypothesis H_0 given as (4) or (6) and the alternative hypothesis H_1 given as (5) and (7), such decisions can be on Wald (1945: 123 and next) and Iwasiewicz (1985: 189):

- accepting hypothesis H_0,
- rejecting hypothesis H_0 and accepting hypothesis $H_{1,}$
- postponing the decision until bringing another unit (sample *n*) to sample *m*.

In this view, sample *n* is the sum of all samples collected in the course of subsequent steps *k*, that is,

$$n = n_1 + n_2 + \ldots + n_k. \tag{10}$$

If for every *k*-th sampling interval the sample size equals one, then the size of the whole cumulative sample is *k* ($n = k$).

In every stage of the tests, the value of the sequential probability ratio test was computed (Iwasiewicz et al. 1988: 92):

$$Q_k = \frac{\prod_{i=1}^{n} \lambda_1 e^{(-\lambda_1 t_i)}}{\prod_{i=1}^{n} \lambda_0 e^{(-\lambda_0 t_i)}}, \tag{11}$$

where: $\lambda_1 e^{(-\lambda_1 t_i)}$ denotes the density function of variable T exponentially distributed, at point t_i if $\lambda = \lambda_1$, while $\lambda_0 e^{(-\lambda_0 t_i)}$ is the value of density function if $\lambda = \lambda_0$.

Each time while estimating the value of test (11), it was tested whether (cf. Wald 1945: 128-129; Iwasiewicz 1985: 189):

$$A < Q_k < B, \tag{12}$$

$$A = \frac{\beta}{1-\alpha}, \text{ whereas } B = \frac{1-\beta}{\alpha}.$$

If $Q_k \le A$, then hypothesis H_0 is accepted with the probability error not above β. If $Q_k \ge B$,

[3] While working on this section, it was mainly based on the works of the Cracow science center, amongst which the following should be mentioned: Iwasiewicz et al. (1988), Iwasiewicz (1985: 187-219; 1999: 243-268; 2001), Major (1997, 2015a, 2015b, 2016).

then hypothesis H_1 should be accepted with the probability of error that is not above α. Moreover, if $A < Q_k < B$, then there is no basis for making either of the decisions mentioned above and tests should be continued.

The inequality (12) can be converted into the following form[4]:

$$z_{g,n} = a + cn > z_n > z_{d,n} = b + cn, \qquad (13)$$

where

$$z_n = \sum_{i=1}^{n} t_i \qquad (n = 1, 2, 3, \ldots; i = 1, 2, 3, \ldots, n) \qquad (14)$$

$$a = \frac{\ln \frac{\beta}{1-\alpha}}{\lambda_0 - \lambda_1}, \qquad (15)$$

$$b = \frac{\ln \frac{1-\beta}{\alpha}}{\lambda_0 - \lambda_1}, \qquad (16)$$

$$c = \frac{\ln \frac{\lambda_0}{\lambda_1}}{\lambda_0 - \lambda_1}. \qquad (17)$$

If in the process of the hypotheses testing a graphic form is employed, then the statistic values (14) computed according to:

$$i = 1: z_1 = t_1;\ i = 2: z_2 = z_1 + t_2;\ \ldots;\ i = n: z_n = z_{n-1} + t_n, \qquad (18)$$

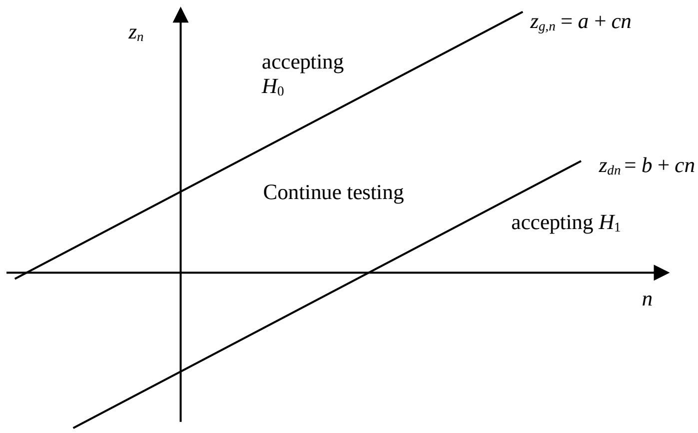
are displayed on an overview diagram (chart), shown schematically in Figure 1.

If at any step of the tests the inequality is such that $z_n \le z_{d,n}$ – then hypothesis H_1 is accepted, with the probability that hypothesis H_0 holds true not exceeding α. If an inequality is such that $z_n \ge z_{g,n}$ – then hypothesis H_0 is accepted, and the probability that hypothesis H_1 holds true does not exceed β.

If, on the other hand, $z_{g,n} > z_n > z_{d,n}$, then there are no grounds to accept any of the hypotheses and testing should be continued, the sample size increased by one with parameters $z_{d,n}$, $z_{g,n}$ and z_n being calculated once again.

[4] The detailed process of transformation can be traced in Iwasiewicz et al. (1988: 92-93).

Figure 1. An overview diagram of a classical sequential procedure.



Source: self-reported data based on Wald (1945), Iwasiewicz et al. (1988: 73).

4. Cumulative sum procedure

4.1 Cumulative sum standard control chart

The term a standard control chart is to be understood as a chart type which does not make it possible to claim that the null hypothesis holds true. On the basis of this chart, one cannot state clearly that the product or technological process tested is reliable. While constructing a standard procedure of cumulative sums the assumption is that the risk of Type II error $\beta = 0$. The implication of this assumption is that the region within which the test is continued is combined with the region resulting from accepting hypothesis $H_{0,}$ Thus, the number of possible decisions is reduced to two. The entire technique involved in the procedure is based on the assumption that the cumulative sum procedure is a classical backward sequential procedure. This fact makes it necessary to plot, at every point ending a particular observation sequence, an ancillary coordinate system turned by 180^{o} in relation to the original one. In every n-th step of the procedure, it is tested whether the so far observed sequence $z_n = \sum_{i=1}^{n} t_i$ is sufficient to accept hypothesis H_1 (finding the product/process failure).

For this method to have greater convenience, one constructs a so called mobile mask, which is then moved across the chart with the increase in the length of the sequence examined. An example of such a mask is presented in Figure 2.

In order to construct the mask, several basic parameters need to be known. These include edges d, h and angle φ, which is an inclination angle of an active edge of the mask to the abscissa of the coordinate system turned by 180^o.

The value of parameter d is determined by calculating the zero of the equation for the lower control line z_d, assuming that β = 0. Considering the fact that the parameters of the mask are determined in the coordinate system that is turned by 180^o, one needs to write the following (see Iwasiewicz et al. 1988: 97; cf. Rao 2013: 231):

$$d = -n = -\frac{\ln\alpha}{\ln\lambda_0 - \ln\lambda_1}. \tag{19}$$

The second of the mask parameters, the angle of inclination of the active edge of the mask, can be determined from the following formula:

$$\varphi = arctgc = arctg\frac{\ln\frac{\lambda_0}{\lambda_1}}{\lambda_0 - \lambda_1}. \tag{20}$$

The last of the parameters is determined from the formula:

$$h = dtg\varphi = dc. \tag{21}$$

Crucial for the mask is its active control edge (BC line in Fig. 3) and point D overlapping with the last point of the sequence under observation. Also, another important rule is that the edge CD should remain parallel to abscissa n.

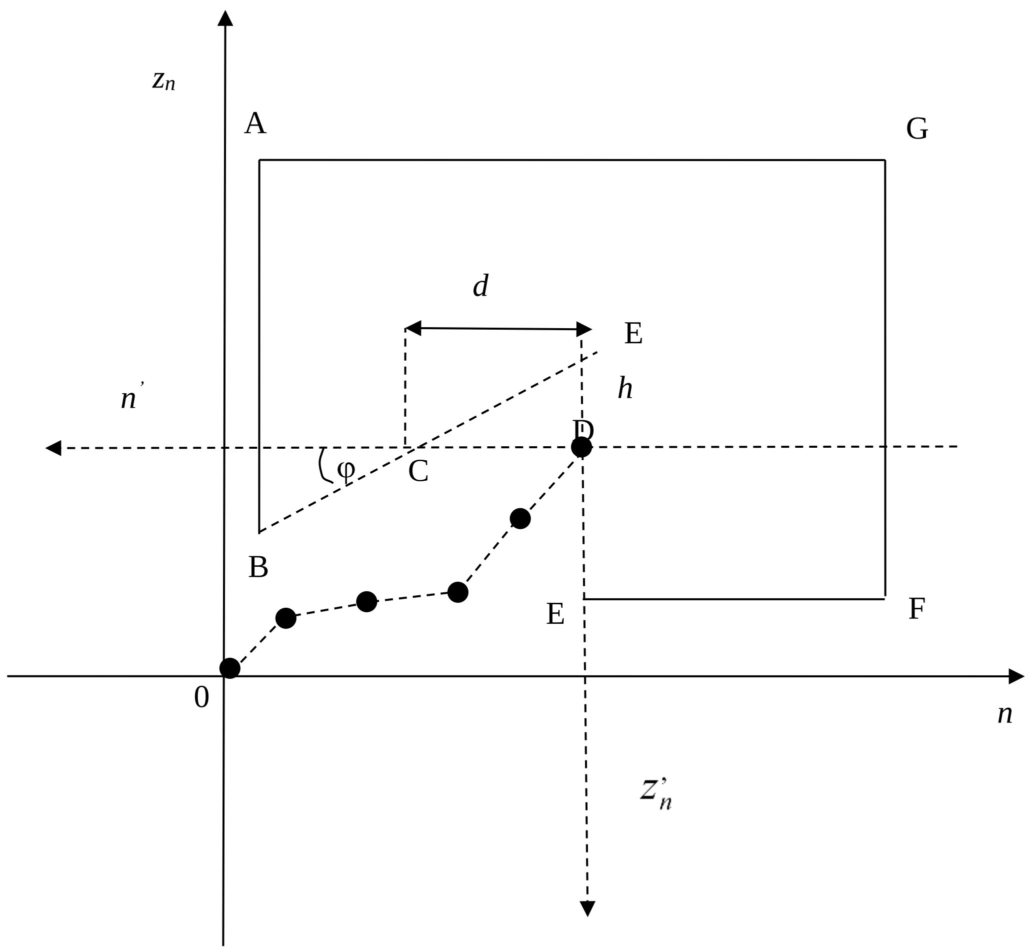
The rules of the procedure are as follows (cf. Major 2015: 35): If at least one of the points in sequence $z_0, \ldots, z_n$ is above the active control edge (line) BC or its extension, then one should accept hypothesis H_1 with an error risk not exceeding α (process/product is faulty). If none of the sequence points is above the edge BC, then one should proceed to the next stage increasing the cumulative sample by the next value t_i.

The application of the mask may at times be quite cumbersome, in particular, when its parameters are expressed by large values. This problem can be prevented by substituting a graphic algorithm with the numerical one. Then the statistic z_n should be replaced with the statistic z_n^*, defined by the following formula (Iwasiewicz 2008-2009: 86):

$$z_n^* = \sum_{i=1}^{n}\left(t_{ji} - c\right) \qquad (n = 1, 2, 3, \ldots) \tag{22}$$

where: $c = tg\varphi$, *j*- a current index functioning over the entire cycle of testing, *i* – operating index functioning within the sequence that is being observed, *n* – the largest value, at a given moment, of the operating index (*i* = 1, 2, 3, …).

Figure 2. Example of the mask applied in the control diagram with the left sided limit to the tolerance interval



Source: Self-reported data based on Iwasiewicz et al. (1988: 75), Rao (2013: 230).

Double index *ji*, in formula (22), demonstrates that the tests are two-stage tests. Index j functions over the entire duration of the tests, while index *i* is initiated when t_i - c < 0. This is a necessary condition for starting to calculate the statistics value (22). Computing the value of the characteristic (22) is continued until one of the conditions shown below is fulfilled:

1° $z_n^* \leq z_d$, where $z_d = dc = h$ (23)

$2^o \; z_n^* \geq 0$ (24)

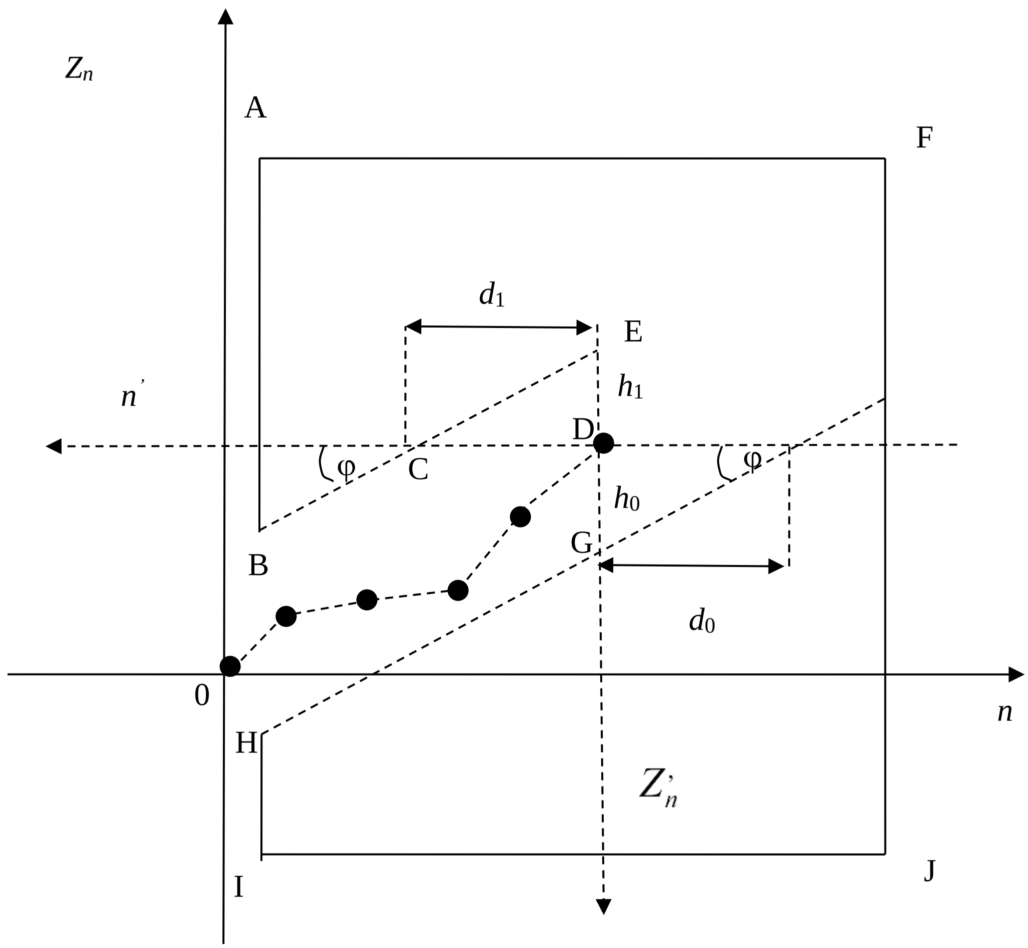
If the condition 1^o is fulfilled, the sequence examined ends with the decision to accept alternative hypothesis H_1 with the risk of error not exceeding α. With the condition 2^o being fulfilled, the computation of the value for characteristic z_n^* is discontinued leading to the zeroing of index i. The sequence of accumulation is resumed when the condition: $t_i - c < 0$ is fulfilled again.

4.2 A modified cusum control chart

The method presented in this section represents the author's modification proposal of the standard cusum control chart outlined in point 4.1, while drawing on the author's earlier works (Major 1997, 2015a, 2015b, 2016). The classical control chart would not allow hypothesis H_0 to be accepted, and thus to claim that the process/product in question is reliable. By having assumed that the risk $\beta = 0$, the region of hypothesis H_0 acceptance is combined with the region where tests are continued. Accepting this hypothesis is made possible through the modified cusum control charts. These charts give up on combining the acceptance regions with those of continued tests. In practical terms, this means that one has to adopt the assumption that $\beta > 0$ and determine anew – as in point 4.1. – the parameters of the modified mask. The illustration of mask is presented in Figure 3.

The new mask has not one but two active edges (control lines). Positive crossing of the upper line (BC) suggests that it is necessary to accept H_1 (the process is unreliable) with the risk of error not greater than α. When there is a negative crossing of the mask lower line, then hypothesis H_0 (the process is reliable) is accepted, with the risk of error being not above β. If, on the other hand, the points in the cumulative sequence fall within the corridor between the active edges of the mask, then the tests are continued, while increasing the sample by a further value t_i. The parameters of the mask are determined according to the rules similar to those found in the classical cumulative sum procedure, with the difference, however, being that the value of parameter β is above zero. Now, in order to be able to build the mask, we need (see Figure 3) the following parameters: d_0, d_1, h_0, h_1, and $c = tg\varphi$.

Figure 3. A modified mask in the control diagram with the left sided limit to the tolerance level



Source: self-reported data based on Iwasiewicz (2008-2009: 87), Iwasiewicz (2011: 241).

Parameter d_0 is obtained as a result of transforming the equation of the lower control line $z_{d,n}$ of the classical sequential procedure (see Fig. 1). The transformation involves determining the zero n_1 of the function $z_{d,n}$. Bearing in mind that the mask functions in a coordinate system that is turned by 180°, we can write (cf. Iwasiewicz 2011: 242-243):

$$d_1 = -n_1 = \frac{\ln(1-\beta) - \ln\alpha}{\ln\lambda_0 - \ln\lambda_1}. \tag{25}$$

The equation of parameter d_0 is determined in a similar way, but this time by transforming the upper control line $z_{g,n}$ and determining the zero n_0. The value of parameter d_1 will then be given by:

$$d_0 = -n_0 = \frac{\ln\beta - \ln(1-\alpha)}{\ln\lambda_0 - \ln\lambda_1}. \quad (26)$$

The value of parameter φ is determined from formula (20), while parameters h_0 and h_1 come from the following formulas:

$$h_0 = d_0 tg\varphi = d_0 c\,„ \quad (27)$$

$$h_1 = d_1 tg\varphi = d_1 c\,. \quad (28)$$

The values of parameters h_0, h_1 and c are also used in the application of a numerical algorithm for testing the process status. The algorithm proceeds then just like in the standard cusum procedure. The difference is that two different sequences can be distinguished in the process, with one leading to *process acceptance* (accepting hypothesis H_0) and the other to its *disqualification (*accepting hypothesis H_1). The process acceptance denotes that the process/product is reliable, while its disqualification is equivalent to considering the process/product to be unreliable. Supposing the point sequence bringing about the process acceptance is called "*sequence A*", while the one resulting in its disqualification is "*sequence B*" (cf. Major 2016: 53-54). In both cases, the cumulative value of statistic z_n^* given by (22) is tested. Not unlike before, index *j* in formula (22) functions throughout the entire duration of tests, with the moment of the index *i* initiation being dependent on the type of the sequence of points.

The analysis begins with tracking the subtraction sign $t_i - c$. When $t_i - c > 0$, counter *i* is launched and "*sequence A*" begins. That the above inequality is fulfilled is a necessary condition for starting the computation of the value of the statistic (22). This calculation is continued until one of the conditions below is met:

1^A $z_n^* \geq z_g^*$, where $z_g^* = d_0 c = h_0$ (29)

2^A $z_n^* \leq 0$ (30)

When the condition 1^A (29) is met, the test ends with the acceptance of the null hypothesis H_0 with the risk of error that is not above β; moreover, when the condition 2^A (30) is met, the calculation of the value of characteristic z_n^* is interrupted and one goes back to tracking the subtraction $t_i - c$. The fulfillment of the conditions (29) and (30) also involves having to zero index *i* ($i = 0$).

Index *i* can also be initiated when $t_i - c < 0$. This is then the beginning of "sequence B", and at the same time the moment when accumulation begins, according to formula (22). The process of such accumulation goes on until one of the conditions below is met:

1^B $z_n^* \leq z_d^*$, where $z_d^* = d_1 c = h_1$ (31)

2^B $z_n^* \geq 0$ (32)

The fulfillment of condition 1^B (31) brings about the acceptance of alternative hypothesis H_1 with the risk of error not exceeding α. The fulfillment of the second condition, 2^B (32), on the other hand, means that the accumulation must be stopped immediately, and we need to get back to tracking the subtraction sign $t_i - c$. As in the preceding case, fulfilling the conditions 1^B or 2^B results in zeroing index *i*.

5. An application example of the control charts outlined

In testing the reliability of materials and devices, diverse factors are taken into consideration. One of them can be the time over which a particular device operates reliably. As already mentioned in the introduction, the time of reliable operation is determined in many cases by the strength of materials of which the product tested is made. The product that could serve as an example is a towing rope. In testing the quality of towing ropes, a number of parameters is checked in terms of their manufacture compliance with the guidelines included in the road traffic regulations (the tow length, placement of the red flag and the rope color), as well as in terms of compliance with technical requirements set out by the Automotive Industry Institute (PIMOT): WT/008/PIMOT/93 –TOW-ROPES. Safety requirements and methods of testing. The following properties are taken into consideration: the operation manual and the tow strength. While testing the strength of the towing rope, it is assumed that[5]:

- towing ropes used for towing vehicles of up to 1500 kg, should be able to transmit a load of not less than 1200daN without any deformations within 20 seconds.
- towing ropes used for towing vehicles of more than 1500 kg, should be able to transmit a load that is not less than the mass of the vehicle towed, e.g. at the vehicle mass of 2000 kg – the testing load of tow-rope should be not less than 2000daN within 20 seconds. Another crucial issue is the measurement of the ultimate strength at which the rope breaks. In this measurement,

[5] After: http://federacja-konsumentow.org.pl/download/LINKI%202.doc [7.01.2018].

the level is determined of the force applied when the rope breaks. A tow-rope with a hook should be able to sustain a load that is greater than that stated in the WT/008/PIMOT/93 rules and should exceed the figure declared by the manufacturer or distributor in a leaflet attached to the rope.

Further on, it was assumed that the a batch of ropes was tested which had already passed positively the tests in terms of all the previous properties with the exception of ultimate strength. Moreover, supposing that the ultimate strength was a random variable X whose distribution was similar to the exponential distribution[6] with Q_0 and Q_{-1} being the expected values of 1500 and 1200daN, respectively. Supposing further, that the performance tests of the strength of towing ropes, yielded the following results:

Table 1. Ultimate strength of a towing rope before breaking

Number of sample/rope i	1	2	3	4	5	6	7	8	9	10	11
Ultimate strength x_t [daN]	2000	2500	1700	2300	3300	4000	4000	3240	4556	1550	1330

Source: conventional data

The data compared in Table 1 were viewed as a sequential sample from the population exponentially distributed. Hypotheses in the forms (4) and (5) were tested:

$$H_0: Q_T = Q_0 = 1500,$$

$$H_{-1}: Q_T = Q_{-1} = 1200,$$

or equivalent forms (6) and (7):

$$H_0: \lambda = \lambda_0 = 1/1500 = 0.00067,$$

$$H_{-1}: \lambda = \lambda_1 = 1/1200 = 0.00083.$$

For testing the hypotheses, the classical sequential procedure and acceptance cusum procedure were used. In the course of the tests the error risk was assumed to be at $\alpha = \beta = 0.05$.

On the basis of the relationships (15) ÷ (17), the values of parameters a,b and c were determined:

[6] The density function argument is tensile force expressed in dekanewtons [daN], while the function values is the probability that the rope will not break.

$$a = \frac{\ln\frac{0.05}{1-0.05}}{0.00067-0.00083} = 17666.63;\; b = \frac{\ln\frac{1-0.05}{0.05}}{0.00067-0.00083} = -17666.63;$$

$$c = \frac{\ln\frac{0.00067}{0.00083}}{0.00067-0.00083} = 1338.9.$$

Thus, inequality (13) was given by:

$z_{g,n}$ = 17666.63 + 1338.9n > z_n > $z_{d,n}$ = -1766.63 + 1338.9n.

The results of the analysis of empirical data using a classical sequential procedure are summarized in Table 2.

Table 2. Reliability analysis of tow-ropes using classical sequential procedure.

i	*n*	x_i	z_n	$z_{d,n}$	$z_{g,n}$	*Decision*
1	1	2000	2000	-16327.8	19005.5	Continue testing
2	2	2500	4500	-14988.9	20344.4	Continue testing
3	3	1700	6200	-13650.0	21683.2	Continue testing
4	4	2300	8500	-12311.2	23022.1	Continue testing
5	5	3300	11800	-10972.3	24360.9	Continue testing
6	6	4000	15800	-9633.5	25699.8	Continue testing
7	7	4000	19800	-8294.6	27038.7	Continue testing
8	8	3240	23040	-6955.7	28377.5	Continue testing
9	9	4556	27596	-5616.9	29716.4	Continue testing
10	10	1550	29146	-4278.0	31055.2	Continue testing
11	11	3330	32476	-2939.2	32394.1	Accept H_0

Source: Author's own calculations

The above analysis showed that in the eleventh step of the sequential procedure, with the size of cumulative sample $n=11$, the null hypothesis H_0 should be accepted with the conclusion that the ultimate strength of tow-ropes meets the requirements assumed, and the product batch manufactured was reliable in this respect. The probability that this estimation was false did not exceed $\beta = 0.05$.

A similar analysis can be carried out based on the modified numerical procedure of cumulative sums. Then, the values of parameters d_0, d_1, and h_0, h_1 (see formulas (25) to (28)) need to be computed additionally.

The parameters are, respectively:

$$d_1 = \frac{\ln 0.95 - \ln 0.05}{\ln 0.00067 - \ln 0.00083} = \text{-13.1953}; \; d_0 = \frac{\ln 0.05 - \ln 0.95}{\ln 0.00067 - \ln 0.00083};$$

$$h_0 = d_0 tg\varphi = d_0 c = z_g^* = 17666.63; \; h_1 = d_1 tg\varphi = d_1 c = z_d^* = \text{-17666.63}$$

A duplicate analysis of the data based on the modified cumulative sum procedure is presented in Table 3.

Table 3. Ultimate strength analysis of tow-ropes using a modified cusum procedure

j	*i*	x_{ji}	*n*	$z_i^* = x_{ji} - c$	z_n^*	*Comments*
1	1	1	1	661.14	661.14	$x_j > c$ (start of *sequence A*, $i = 1$)
2	2	0	2	1 161.14	1 822.28	accumulate ($i = i + 1$)
3	3	1	3	361.14	2 183.42	accumulate ($i = i + 1$)
4	4	1	4	961.14	3 144.55	accumulate ($i = i + 1$)
5	5	1	5	1 961.14	5 105.69	accumulate ($i = i + 1$)
6	6	0	6	2 661.14	7 766.83	accumulate ($i = i + 1$)
7	7	1	7	2 661.14	10 427.97	accumulate ($i = i + 1$)
8	8	1	8	1 901.14	12 329.11	accumulate ($i = i + 1$)
9	9	1	9	3 217.14	15 546.25	accumulate ($i = i + 1$)
10	10	0	10	211.14	15 757.39	accumulate ($i = i + 1$)
11	11	0	11	1 991.14	17 748.53	$z_n^* > z_g^*$ (accept H_0)

Source: Author's own calculations

As can be gleaned from the above, the modified cusum procedure also needed eleven steps in order to decide whether to accept the null hypothesis. The cumulative process started in period $t = 1$ and continued with no interruption to period $t = 11$.

6. Conclusions

The example illustrating the application of control charts, as presented in point 5, refers to a particular case in which a test of a destructive nature needs to be carried out during the product reliability testing. The product that has been one-time tested can no longer be used. On the other hand, however, the information on its quality and reliability is of key importance for the safety of its operations. We also face a similar situation while analyzing the safety of operations of mechanical vehicles during so called crash tests. In order to find out whether a particular construction is reliable one has first to destroy the vehicle. As a rule, such tests are expensive and one cannot afford to use large samples or a 100% testing. From the economic perspective, one should seek to apply such control procedures which would bring as maximum as possible reliability probability with an experimental sample that is as small as possible. From the theoretical point of view, classical procedures for testing hypotheses could be used based on significance tests (e.g. Shewhart control charts).

However, their shortcoming is that they will not allow for accepting the null hypothesis with a given probability of Type II error. Moreover, they require that the sample size be predefined at the beginning of testing. Theoretically, the control procedure drawing on Shewhart control charts with the option of the process acceptance could be used. The description of their functioning can be found in the works of e.g. Iwasiewicz (1985: 57-86, 159-163; 1999: 239-242; 2001), Major (2015b) and in PN-ISO 7966, 2001 standard. Yet, they also require initial determination of a necessary sample size needed for one of the hypotheses tested to be accepted. In the sequential procedures and cumulative sum procedures created on their basis, the size of a sample is a parameter specified only at the moment of accepting or disqualifying a process. This brings about that there is no risk of the sample size overestimation, and thus generating unnecessary control costs. Therefore, the sequential methods presented in this paper and used in the evaluation of the production process appear considerably more efficient and cost-effective than the comparable procedures based on Shewhart control charts. In summing up, one can conclude that the sequential methods and the examples illustrating the application of the solutions outlined seem to confirm the thesis advanced in the paper, while the methods presented herein could be used in practice by economists, quality managers or production technologists.

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